Technical Notes

TECHNICAL NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Apparent Subsonic Choking in Dual-Stream Nozzles

E. Fage*
ASTECH, Meudon-la-Forêt, France

Nomenclature

A = cross-sectional area
a = sound speed
p = static pressure
P = total pressure
M = Mach number
R_j = radius of the primary jet
w = mass flow rate
x = distance along the nozzle
γ = specific heat ratio
θ = angle of the supersonic jet boundary relative to nozzle axis
μ = secondary-to-primary mass flow rate ratio

Subscripts
s = secondary (or fan) flow
j = main (or jet) flow

Superscripts
$*$ = refers to conditions for which choking – or generalized choking – occurs

Introduction

DUAL stream propulsive nozzles have already been the subject of numerous theoretical and experimental studies. In order to describe the associated phenomena, a flow model based on one-dimensional studies by J. Fabri, E. le Grieves, and R. Siestrunck[1] is generally used, complemented by a three-dimensional analysis as proposed by W. L. Chow and A. L. Addy.[2] A common assumption is that, under certain conditions, the high-pressure primary jet expands to supersonic speeds while the secondary flow is progressively accelerated up to a sonic throat. This flow is then choked between the nozzle wall and the expansion of the primary jet, and upstream conditions will then be insensitive to an additional reduction of the discharge pressure. Many tests seem to confirm the validity of the above mentioned scheme.

It would appear, however, that certain test results do not conform to this scheme, in so far as the secondary mass flow is observed to remain choked even though the secondary Mach number is much less than 1 (e.g. $M_s \sim 0.5$). This experimental result is consistent with further one-dimensional analyses published in the recent years. For example, Bernstein, Heiser, and Hevenor[3] and Hoge and Segars[4] have discussed the concept of "compound choking" or "generalized choking" which could explain the previous experimental finding. But since these analysis are valid for only one-dimensional flows, generally they have been overlooked or thought to be of limited practical interest.

This Note outlines a different one-dimensional approach which can readily be extended to a three-dimensional study. Application of the theory to specific cases confirms the existence of "generalized "generalized choking" and shows very good agreement between predicted and measured three-dimensional choked configurations.

One-Dimensional Case

First of all, let us consider the one-dimensional flow through an arbitrarily shaped ejector. We are particularly interested in the operating conditions known as "supersonic" or choked,[1,2] which are established when the primary stream at high pressure chokes – by its supersonic expansion – the lower pressure secondary flow. Choking here signifies that the total flow rate through the ejector is the maximum possible, consistent with the total pressure P_s. (The primary mass flow is assumed to be permanently choked at the throat of the primary nozzle). Many researchers have formulated that this implies that a maximum secondary speed, corresponding to $M_s^* = 1$, is reached in the plane where choking occurs. This would be an expansion of the classical result valid for a solid wall convergent nozzle.

However, it is essential to note here that the subsonic secondary flow is not accelerated through a solid wall convergent nozzle. For a given total secondary pressure P_s, a mass flow increase requires a greater M_s and a lower p_s (at any station). Consequently, the supersonic primary flow must expand further, and the fluid boundary moves so as to reduce the secondary channel, the area of which is thus a function $A_s = A_s(p,x)$. Let us consider more closely the prevailing expansion

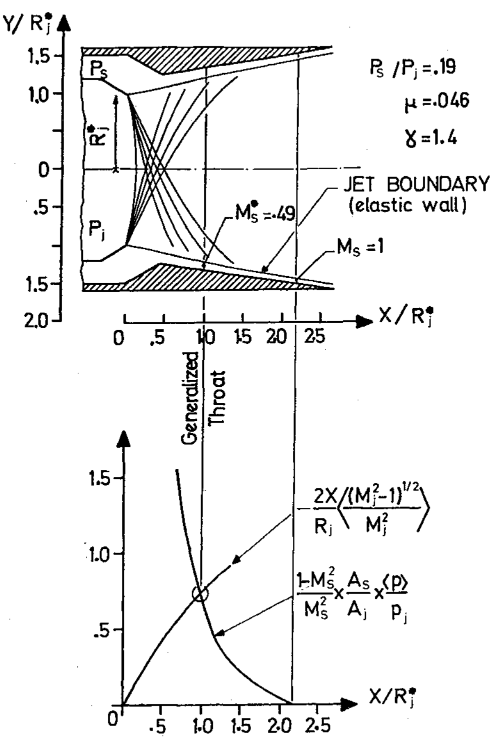

Fig. 1 Method for approximate location of the generalized throat in a dual stream nozzle.

Presented as Paper 74-1176 at the AIAA/SAE 10th Propulsion Conference, San Diego, Calif., October 21-23, 1974; submitted November 21, 1974; revision received December 15, 1975. The author gratefully acknowledges the basic participation of R. Marchal and P. Servanty in clarifying and formulating the analysis reported in the present Note.

Index category: Jets, Wakes, and Viscid – Inviscid Flow Interactions.

*Technical Director. Member AIAA.

conditions for a compressible fluid inside such a flexible nozzle.

At any particular station, the area is a function of the internal pressure only, and the derivative dA/dp can be called the "local nozzle elasticity." For $dA/dp > 0$, an increase in secondary speed (i.e., a drop in secondary pressure) is then somewhat offset by a reduction in secondary area and a situation may arise where both effects exactly cancel each other, leaving the mass flow constant. Under such conditions, corresponding to choking, no further mass flow increase would be possible.

A well-known relationship for constant mass flow (using Shapiro's one-dimensional differential equation[5]) is

$$\frac{dA}{A} = \frac{1-M_s^2}{\gamma_s M_s^2}\frac{dp}{p} \tag{1a}$$

where dA/A and dp/p are the logarithmic derivatives corresponding to an infinitesimal displacement dx along a frozen stream tube, within a solid wall nozzle. Equation (1), in this context, expresses the pressure change dp/p, resulting from the area change dA/A imposed by the nozzle shape.

But, for a flexible nozzle, a pressure change dp/p created by some external change in the boundary conditions will impose, at any fixed station, an area change dA/A and, generally some mass flow variation. In this completely different context, Eq. (1) defines the local Mach number for which both changes dp/p and dA/A result in a stationary, i.e., maximum, mass flow.

The fundamental role played by the elasticity of the fluid boundary is better shown by re-writing Eq. (1) as

$$\left(\frac{dA}{dp}\right)_s = \frac{1-M_s^2}{\gamma_s M_s^2}\frac{A_s}{p_s} \tag{1b}$$

Immediately, it can be seen that, except for the case of a perfectly rigid wall, for which $dA/dp = 0$ everywhere, maximum flow will not be reached for $M_s = 1$, but for some subsonic value varying with the local wall elasticity, dA/dp.

In the general case under consideration here, the elasticity of the fluid boundary is defined by the coexistence of the two streams within a given total cross-section which imposes

$$\left(\frac{dA}{dp}\right)_s + \left(\frac{dA}{dp}\right)_j = 0 \tag{2}$$

Again, the derivatives dA/dp are taken at any fixed station and Eq. (2) is valid irrespective of the shape of the ejector. It merely states that stream tube elasticities must be equal and opposite in signs, so that the effect of a pressure change dp is limited to a displacement of their fluid partition, with no effect on the overall cross section.

As the primary flow is supposedly choked at its own throat, we have at any cross-section of the primary stream, expanded from P_j

$$\left(\frac{dA}{dp}\right)_j = \frac{1-M_j^2}{\gamma M_j^2}\frac{A_j}{p_j} \tag{3}$$

The relationship defining the critical secondary Mach number M_s^* is then obtained by combining Eqs. (1b), (2), and (3) and, since $p_s = p_J$ we get

$$\frac{A_s}{\gamma_s}\frac{1-M_s^{*2}}{M_s^{*2}} + \frac{A_j}{\gamma_j}\frac{1-M_j^2}{M_j^2} = 0 \tag{4}$$

Since M_s^* is subsonic it could be said that the maximum secondary mass flow is controlled by "subsonic" choking. However, this situation can only occur through the interaction of the adjacent supersonic stream, and is probably better described as "compound," or "generalized" choking.

Although Eq. (4) is formally identical to condition $\beta = 0$ in Ref. 3, there is a basic difference in the way it is obtained. Bernstein, Heiser, and Hevenor had to assume that $(dA/dx)_s + (dA/dx)_j = 0$ which is valid only at the geometrical throat. In the present Note, Eq. (4) results from $(dA/dp)_s + (dA/dp)_j = 0$ which holds at any station along the nozzle. It follows that the present approach can be extended to cases for which choking does not necessarily occur at the geometrical throat.

Three-Dimensional Extension

In the general case, the primary jet displays highly three-dimensional trends and $(dA/dp)_j$ is not correctly defined by Eq. (3). However, one can assume a one-dimensional secondary flow, and Eqs. (1b) and (2) are still valid. Therefore, provided an adequate expression was derived for $(dA/dp)_j$, the previous one-dimensional analysis could be repeated, and the choking condition will again be found through

$$(dA/dp)_j = -(dA/dp)_s = -\frac{1-M_s^2}{\gamma M_s^2}\frac{A_s}{p_s}$$

If a method of characteristics is used to study the conditions of co-existence of the primary and secondary streams, it becomes obvious that the primary jet cross-section is no longer a function of the local boundary static pressure only. The latter simply controls the slope θ of the fluid boundary, and the jet elasticity can be written

$$\left(\frac{dA}{dp}\right)_j = 2\pi R_j \frac{dR}{dp} = 2\pi R_j \int_0^x \frac{d\theta}{dp} dx$$

For infinitesimal dp values,

$$\frac{d\theta}{dp} = -\frac{(M_j^2-1)^{1/2}}{p\gamma M_j^2}$$

M_j is the Mach number taken on the primary side of the interface, where $p_j = p_s$. Hence

$$\left(\frac{dA}{dp}\right)_j = -2\pi R_j \int_0^x \frac{(M_j^2-1)^{1/2}}{\gamma p M_j^2} dx \tag{5}$$

Combining Eqs. (1b), (2) and (5) again yields M_j and M_s^* for generalized choking

$$2\pi R_j \int_0^x \frac{(M_j^2-1)^{1/2}}{\gamma p M_j^2} dx = \frac{1-M^{*2}_s}{\gamma M^{*2}_s}\frac{A_s}{p_s} \tag{6}$$

For many practical examples, M_j does not vary rapidly downstream of the primary throat and $(M_j^2-1)^{1/2}M_j^{-2}$ is nearly constant, so that Eq. (6) reduces to

$$2\frac{X}{R_j}\left\langle\frac{(M_j^2-1)^{1/2}}{M_j^2}\right\rangle = \frac{1-M_s^{*2}}{M_s^{*2}}\frac{A_s}{A_j}\frac{\langle p\rangle}{p_s} \tag{7}$$

Figure 1 shows how Eq. 7 can be graphically solved on a practical example.

Equation (7) has been checked against many test results, and the agreement between predicted and actual "generalized throat" has been satisfactory. For example, experimentally it has been verified that strong perturbations ahead of the secondary sonic throat would not unchoke the nozzle, unless they come immediately aft of the generalized throat. Thus, it can be said that subsonic or generalized choking is not just a one-dimensional paradox, but a real phenomenon. Some practical consequences stemming from the existence of generalized choking are further discussed in Ref. 6.

References

[1] Fabri, J., Grives, E., and Siestrunk, R., "Etude Aerodynamique des Trompes Supersoniques," *G. fur luftfahrt*, Jan. 1953, pp. 101-111.

[2]Chow, W. L. and Addy, A. L., "Interaction between Primary and Secondary Streams of Supersonic Ejector Systems and their Performance Characteristics," *AIAA Journal*, Vol. 2, April 1964, pp. 686-695.

[3]Bernstein, A., Heiser, W. H., and Hevenor, C., "Compound – Compressible Nozzle Flow," *Journal of Applied Mechanics*, Sept. 1967.

[4]Hoge, H. J. and Segars, R. A., "Choked Flow: A Generalization of the Concept and Some Experimental Data," *AIAA Journal*, Vol. 3, Dec. 1965, pp. 2177-2184.

[5]Shapiro, A. H., "The Dynamics and Thermodynamics of Compressible Fluid Flow," Ronald Press, Co., New York, 1953.

[6]Fage, E., "Apparent Subsonic Choking in Dual-Stream Nozzles, and Practical Consequences," AIAA Paper 74-1176, 1974.

Accurate Prediction of Critical Conditions for Shear-Loaded Panels

George J. Simitses* and Izhak Sheinman†
Georgia Institute of Technology, Atlanta, Ga.

GENERAL instability of eccentrically stiffened thin cylindrical panels subjected to uniform axial compression, uniform pressure, and shear has been investigated by Simitses.[1]. The procedure employed is similar to those used in the study of constant-thickness isotropic cylindrical panels.[2-4]

In the case of uniaxial or biaxial compression with classical simply supported boundaries, the solution to the buckling equations is taken as $\sin(m\pi x/L)\sin(n\pi y/b)$ and the critical load is obtained through minimization with respect to the wave numbers m and n. In the case of shear or combination of shear with uniaxial or biaxial compression, because of the nature of the buckling equation, usually a Rayleigh-Ritz or a Galerkin procedure is employed. The transverse displacement is represented by a double Fourier series,[1,2] and the critical condition is obtained from the solution of a determinant (characteristic equation) which is dependent on the number of terms taken in the series representation of the transverse displacement; the larger the size of the determinant, the better will be the approximation. But, as the size of the determinant is increased, the computer time required to calculate critical conditions increases rapidly. It therefore is desirable to keep the size of the determinant as small as posssible without sacrificing accuracy. Schildcrout and Stein[2] report that 10 terms in the series representation usually are sufficient to predict critical conditions accurately. Of course they meant the first 10 terms, and this is questionable, especially in the case of stiffened configurations.

The requirement of keeping the size of the determinant small and thus the computer time required low is extremely important in the optimization of stiffened panels and individual or combined load conditions, which include shear. This is so because, in the process of arriving at the optimum configuration, it is necessary to compute critical conditions (solve the determinant) for numerous combinations of the geometric parameters. The present paper deals with the problem of finding a reasonable computational method (from the point of view of computation time) to compute critical conditions accurately for shear loaded panels. This is accomplished by keeping the size of the determinant low through the retention of the most influencing terms in the Fourier expansion for the transverse displacement. In addition, the effect of the different geometric parameters on the size and type of the needed determinant is investigated.

The governing equations[1] are

$$A_{mn}\alpha_{mn} + k_s \sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty} \frac{A_{m'n'}\, mn\, m'n'}{[(m')^2 - m^2]\,[(n')^2 - n^2]} = 0 \qquad (1)$$

for $m,n = 1,2$ where $m \pm m' = $ odd, $n \pm n' = $ odd, and where

$$\alpha_{mn} = (\pi^2/32\xi)\,[p(m,n) - k_x m^2 - k_y \xi^2 n^2] \qquad (2)$$

where ξ is the panel aspect ratio (L/b), p is a function of m,n and the geometric parameters, and k_x, k_y, k_s are the load parameters.[5] Equation (1) can be decoupled into two systems, one corresponding to $m \pm n = $ even and one $m \pm n$ odd. The former is referred to as symmetric buckling and the latter as antisymmetric buckling. Both characteristic equations (determinants) must be solved in order to find the critical condition for a given set of geometric parameters.

In general, each one of the two determinants, one for symmetric buckling and one for antisymmetric bucking, can be written in the following matrix form:

$$\{[A] + k_s[B]\}\,\{X\} = 0 \qquad (3)$$

where $\{X\}$ is the displacement vector that consists of the Fourier coefficients A_{mn}, $[A]$ is a diagonal matrix containing α_{mn} only, and $[B]$ is a matrix that includes all of the remaining terms and has zero elements along the diagonal. Since both matrices are symmetric and since matrix $[A]$ is positive definite (this is definitely true for shear loaded panels, but it is true also for the case of combined shear with other destabilizing loads, k_x and k_y, provided that the panel does not buckle when $k_s = 0$), then the eigenvalue k_s is always real.[7]

The eigenvalue k_s and the corresponding eigenvector $\{X\}$ can be calculated by matrix iteration. Since the sign of the eigenvalue does not affect the solution, it is calculated by employing a Rayleigh quotient[6] in the form

$$k^2_{s_{j+1}} = \{V_{2j+2}\}^T\{V_{2j}\}/\{V_{2j+2}\}^T\{V_{2j+2}\} \qquad (4)$$

where $j = 0,1,2,\ldots$(iteration number), and $\{V_{j+1}\}$ is calculated from the equation

$$[A]\{V_{j+1}\} = -[B]\{V_j\} \qquad (5)$$

by arbitrarily choosing the initial vector $\{V_0\}$. The corresponding eigenvectors are given by

$$\{X_1\}_{j+1} = \{V_{2j+1}\} + k_{s_{j+1}}\{V_{2j}\} \quad \text{for positive } k_s \qquad (6a)$$

$$\{X_2\}_{j+1} = \{V_{2j+1}\} - k_{s_{j+1}}\{V_{2j}\} \quad \text{for negative } k_s \qquad (6b)$$

This iteration procedure is continued until the change in the computed eigenvalue, δk_s, from two successive iterations, is smaller than a specified number ϵ (convergence criterion). Through this well-known procedure, the desired accuracy for the eigenvalues is achieved, but through the corresponding eigenvector one easily can assess its most influencing elements (A_{mn}).

It is conjectured that, if the the most influencing term in the eigenvector is $A_{k\ell}$, then 1) as the subscripts m and n move away from k and ℓ the relative magnitude of A_{mn} diminishes, and 2) if all of the A_{mn} terms with relative magnitudes smaller than 10% of the magnitude of $A_{k\ell}$ are excluded from the

Received April 17, 1975; revision received Jan. 21, 1976. This research was sponsored by the Air Force Office of Scientific Research, Air Force Systems Command, U.S. Air Force, under AFOSR Grant 74-2655.

Index categories: Structural Stability Analysis; Aircraft Structural Design (including loads).

*Professor, School of Engineering Science and Mechanics. Associate Fellow AIAA.

†Postdoctoral Fellow, School of Engineering Science and Mechanics; on leave from Technion-Israel Institute of Technology, Haifa, Israel.